Tutorial 5

In the following problems, V denotes a finite-dimensional inner product space.

1. Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$ is normal. Prove that T is a projection if and only if the eigenvalues of T are contained in $\{0, 1\}$.

2. Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$ is normal with all real eigenvalues. Show that T is self-adjoint.

3. Let $T_1, T_2 \in \mathcal{L}(V)$ be self-adjoint operators with identical eigenvalues and multiplicities. Show there exists an isometry $S \in \mathcal{L}(V)$ such that $T_1 = S^* T_2 S$.

4. Suppose $T \in \mathcal{L}(V)$ is positive and $T^2(v) = v$. Prove that $T(v) = v$.

5. Let V be the inner product space of real-valued continuous functions on $[0, 1]$ with inner product

   $$\langle f, g\rangle = \int_0^1 f(x)g(x)\ dx$$

   Given fixed $p \in V$ define $T_p \in \mathcal{L}(V)$ as $T_p(f) = pf$.

   (a) For which functions p is T_p a positive operator?

   (b) For which functions p is T_p an isometry?

6. Suppose $\mathbb{F} = \mathbb{C}$. Prove $T \in \mathcal{L}(V)$ is normal if and only if $T^* = f(T)$ for some polynomial f.

7. A linear operator $T \in \mathcal{L}(V)$ is a *kinda-isometry* if there exists a subspace $U \subseteq V$ such that $\langle T(u_1), T(u_2)\rangle = \langle u_1, u_2\rangle$ for all $u_1, u_2 \in U$ and $T(v) = 0$ for all $v \in U^\perp$.

   (a) Why is this definition different from saying "$T|_U$ is an isometry on U and $T|_{U^\perp} = 0$"?

   (b) Show that if T is a kinda-isometry then $T^*T = P_U$.

8. A *translation* on V is a function $t\colon V \to V$ of the form

   $$t(x) = x + v$$

   where $v \in V$ is some fixed element. A *rigid transformation* on V is a function $r\colon V \to V$ such that for all $x, y \in V$,

   $$\|r(x) - r(y)\| = \|x - y\|$$

   Note that translations and isometries are rigid transformations. Classify all rigid transformations when $\mathbb{F} = \mathbb{R}$.